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Exploratory Approach to Explicit Solution of Nonlinear Evolution Equations

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By introducing a concept of "rank," an exploratory approach is presented for finding the explicit traveling wave solutions to some physically interesting nonlinear evolution equations in physics and other fields. To show the efficiency of this approach, the explicit traveling wave solutions to some well-known equations, such as the MKdV–Burgers equation, the generalized Fitzhugh–Nagumo equation, the generalized Burgers–Fisher equation, and some coupled ones in fluid mechanics, are given.

1. INTRODUCTION

Many dynamic problems in physics and other fields are usually characterized by nonlinear evolution partial differential equations (NLEPDEs), which are often called governing equations. For example, disturbance or wave propagation problems in fluid mechanics, space plasma physics, atmospheric science, etc., are such cases. To understand the physical mechanism of these problems one has to study the solutions to the associated governing equations. Looking for analytical solutions to nonlinear physical models has long been a major concern for both mathematicians and physicists since they can provide much physical information and more insight into the physical aspects of the problem and thus lead to further applications. Much work has been done over the years on the subject of obtaining explicit solutions to nonlinear partial differential equations of various nonlinear physical phenomena. Although many efficient approaches have been proposed (Coffey, 1990, 1992; Feng, 1996, 1998, 1999a, b; Hereman *et al.*, 1986; Hereman and Takaoka, 1990; Huang *et al.*, 1989; B. L. Lu *et al.*, 1993a, b; B. Q. Lu, 1994; Parkes

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et al., 1998; Sachev, 1987; M. L. Wang, 1996; M. L. Wang *et al.*, 1996; Yang, 1994, 1995; and references therein), there is no operational method available for getting the explicit solutions to a given nonlinear physical model.

The aim of the present paper is to tentatively explore an operational method by introducing the concept of "rank" for the derivatives involved. The paper is organized in as follows: In the next section we propose an exploratory approach for finding explicit solitary wave solutions of some physical nonlinear evolution models; then we give examples to illustrate the efficiency of this approach; we end with some conclusions.

2. SKETCH OF EXPLORATORY APPROACH

We consider the traveling wave solution $u(x, t) = u(\xi)$, $\xi = x - \lambda t$, and study a general nonlinear equation of the form

$$\begin{aligned} & f_1(u)u_t + f_2(u)u_x + f_3(u)u_{tt} + f_4(u)u_{tx} + f_5(u)u_{xx} \\ & \quad + f_6(u)u_t^2 + f_7(u)u_tu_x + f_8(u)u_x^2 \\ & \quad + f_9(u)u_{ttt} + f_{10}(u)u_{ttx} + f_{11}(u)u_{txx} + f_{12}(u)u_{xxx} \\ & \quad + f_{13}(u)u_tu_{tt} + f_{14}(u)u_{tt}u_x + f_{15}(u)u_tu_{xx} \\ & \quad + f_{16}(u)u_xu_{xx} + f_{17}(u)u_t^3 + f_{18}(u)u_t^2u_x + f_{19}(u)u_tu_x^2 + f_{20}(u)u_x^3 \\ & \quad + \ldots = g(u) \end{aligned} \tag{1}$$

where $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, $f_i(u)$, $i = 1, 2, 3, \ldots$, and $g(u)$ are polynomials in u, and λ is a undetermined real constant. Inserting $u(x, t) = u(\xi)$ into (1), we have

$$\begin{aligned} & [-\lambda f_1 + f_2]u' + [\lambda^2 f_2 - \lambda f_4 + f_5]u'' + [\lambda^2 f_6 - \lambda f_7 + f_8]u'^2 \\ & \quad + [-\lambda^3 f_9 + \lambda^2 f_{10} - \lambda f_{11} + f_{12}]\, u''' + [-\lambda^3 f_{13} + \lambda^2 f_{14} - \lambda f_{15} + f_{16}]u'u'' \\ & \quad + [\lambda^3 f_{17} + \lambda^2 f_{18} - \lambda f_{19} + f_{20}]u'^3 + \ldots = g(u) \end{aligned} \tag{2}$$

We call Eq. (2) the reduced equation of (1) if any further direct integration of (2) is impossible, the meaning of which will become clear after reading the examples given below.

In order to introduce the exploratory solution approach, let us define the concept of "rank." If the nonlinear term in the reduced ordinary differential equations can be written as

$$u^{k_0}u'^{k_1}(u'')^{k_2} \ldots (u^{(m)})^{k_m}$$

with k_j real constants, then the rank of this term is defined as the number

$$0k_0 + k_1 + 2k_2 + \dots + mk_m$$

that is, by the sum of the number of $d/d\xi$. If the rank of every term in the reduced ordinary differential equation is even, the candidate ansatz may be

$$u'' = p(u)$$

or

$$u = f(v, v')\left(\sum_{i=0}^{k} c_i v^i\right), \qquad v'' = p(v) \tag{3}$$

In this case, bounded bell-shaped solutions often appear. If otherwise, the candidate ansatz may be

$$u' = p(u)$$

or

$$u = f(v)\left(\sum_{i=0}^{k} c_i v^i\right), \qquad v' = p(v) \tag{4}$$

In this case, bounded solutions are usually kinks. This kind of ansatz usually could be made to obtain the exact solutions to reaction-diffusion equations, such as done, say, in Lu *et al.* (1993a, 1993b), M. X. Wang *et al.* (1994), and Feng (1996) and references therein. But the concrete form of the ansatz depends on balancing the dominant nonlinear terms in the reduced ODE.

The ansatz method seems to be a powerful tool for dealing with coupled nonlinear physical models. For a coupled system of nonlinear differential equations with two unknowns

$$\begin{aligned} &F_1(u, v, u_t, v_t, u_x, v_x, u_{xx}, v_{xx}, \dots) = 0 \\ &F_2(u, v, u_t, v_t, u_x, v_x, u_{xx}, v_{xx}, \dots) = 0 \end{aligned} \tag{5}$$

the ansatz method sometimes works. As for the traveling wave solutions to (3), we have to solve its corresponding reduced ordinary differential equations

$$G_1(u, v, u', v', u'', v'', \dots) = 0, \qquad G_2(u, v, u', v', u'', v'', \dots) = 0 \tag{6}$$

It is obvious that to solve (6) exactly is more difficult than solving (2). The exact solvability of (6) often involves two kinds of ansätze: one relation between the two unknowns u and v, and another ansatz for one of them. Generally speaking, when a coupled system of two differential equations is considered, a delicate explicit assumption between the two unknowns will decouple the system; a suitable ansatz for one unknown, combined with the assumed relation between them, will solve the problem if we are lucky. Thus, the difficulty for solving a coupled differential equation system lies in finding the explicit relation between the two unknowns or their derivatives and one ansatz for one of them or suitable explicit parameter expressions for them such as $u = f(w, w', w'', \ldots)$, $v = g(w, w', w'', \ldots)$, and one ansatz for w, such as done in Lan and Wang (1990), B. L. Lu *et al.* (1993b), M. X. Wang *et al.* (1994), and Feng (1996, 1999a, b). Principally, the concrete ansatz chosen for specific coupled nonlinear equations depends on the topological properties of the equations themselves and must enable us to solve one of them in closed form. Coupled nonlinear differential equations possess richer solutions than single ones, but it is difficult to determine what kind of ansatz to use for a given class of coupled nonlinear equations, as pointed out by Feng (1996).

After inserting the given ansatz into the reduced ordinary differential equation, a set of algebraic equations is often obtained by balancing the dominant nonlinear terms. The solutions to this set of algebraic equations give us various relations among the physical parameters and the undetermined constants in the ansatz. Usually, the corresponding algebraic equations can be solved manually. But sometimes it becomes so complex that it presents us another challenge. So some scientists resort to, say, *Mathematica*, as done in Feng (1998, 1999a, b) and Parkes *et al.* (1998). High-performance computers and specialized software make this kind of work much easier.

In what follows, the efficiency of the exploratory approach stated above is shown by finding exact traveling solutions to some examples of physical nonlinear evolution models.

3. EXAMPLES

The traveling wave solutions of the form $u = u(\xi)$, $\xi = x - \lambda t$, are considered in all the following examples.

Example 1. Ito equation (Ito, 1980):

$$u_t = u_{5x} + 54u^2u_x + 18uu_{3x} + 36u_xu_{xx}$$

Considering its traveling wave solution and integrating once, we see

$$\lambda u + u^{(4)} + 27u^3 + 18uu'' + 9u'^2 = E \tag{1.1}$$

where E is an integration constant. Equation (1.1) is just the so-called reduced equation. Obviously, the rank of every term in (1.1) is even. This suggests that we use an ansatz like (3). To this end, let us make the following ansatz:

$$u = A + v, \qquad v'' = bv^2 + cv \tag{1.2}$$

where A, b, c are undetermined constants. Then,

$$v'^2 = \frac{2b}{3} v^3 + cv^2, \qquad v^{(4)} = \frac{10}{3} b^2 v^3 + 5bcv^2 + c^2 v \tag{1.3}$$

Insert (1.2) and (1.3) into (1.1) and balance the same powers of v in the resulting equation to obtain

$$27 + 24b + \frac{10}{3} b^2 = 0, \qquad 54A + 18\,Ab + 27c + 5bc = 0$$

$$\lambda + 54A^2 + 18Ac + c^2 = 0, \qquad E = \lambda A + 27A^3$$

which gives

$$b = \frac{9}{10}(-4 \pm \sqrt{6}), \qquad c = \frac{6}{5}(-11 \pm 4\sqrt{6})\,A$$

$$\lambda = \frac{18}{25}(-179 \pm 56\sqrt{6})\,A^2, \qquad E = \frac{18}{25}[-141.5 \pm 56\sqrt{6}]\,A^3 \tag{1.4}$$

From (1.2) and (1.4) we know that

$$u = A + v = A - \frac{3c}{2b}\operatorname{sech}^2\left[\frac{\sqrt{c}}{2}(x - \lambda t - \xi_0)\right] \qquad \text{or}$$

$$A + \frac{3c}{2c}\operatorname{csch}^2\left[\frac{\sqrt{c}}{2}(x - \lambda t - \xi_0)\right]$$

with b, c, λ satisfying (1.4). Of course, $c > 0$ is required.

Other kinds of higher order KdV-like equations, such as the fifth- and $(2k + 1)$th odd-derivative-order KdV-related nonlinear equations (Feng, 1998, 1999a, b; Parkes *et al.*, 1998, and references therein), can also be discussed by using the ansatz $u'' = bu^{k+1} + cu$ or $u = p(v)$, $v'' = bv^2 + cv$ with $p(v)$ a polynomial of v, since their corresponding reduced ordinary differential equations are of even order by the rank concept here.

Example 2. The generalized Fithugh–Nagumo equation (Yang, 1994):

$$u_t - \alpha u_{xx} = \beta u(1 - u^\delta)(u^\delta - r)$$

where α, β, $\delta > 0$ and $r \in [-1, 1)$.

Considering the traveling wave solution to the above equation, we have

$$-\lambda u' - \alpha u'' = \beta u(1 - u^\delta)(u^\delta - r) \tag{2.1}$$

It is evident that Eq. (2.1) has both odd- and even-rank terms. According to (4), this suggests we make the ansatz

$$u = v^h, \qquad v' = bv^2 + cv \tag{2.2}$$

where h, b, c are undetermined constants. Then,

$$u' = h(bv^{h+1} + cv^h), \qquad u'' = b^2(h+1)hv^{h+2} + (2h+1)hbcv^{h+1} + c^2h^2v^h \tag{2.3}$$

Insert (2.2) and (2.3) into (2.1) and balance the same powers of v in the resulting equation to obtain that $h = 1/\delta$ and

$$-\lambda hc - \alpha c^2h^2 + \mathrm{r}\beta = 0, \qquad \lambda bh + \alpha(2h+1)hbc + (r+1)\beta = 0$$
$$\alpha b^2(h+1)h + \beta = 0 \tag{2.4}$$

Solving the algebraic equation (2.4) gives us the following four sets of solutions:

$$\lambda = \mp((1+\delta)r - 1)\sqrt{\frac{\alpha\beta}{1+\delta}}, \qquad b = \pm\delta\sqrt{\frac{\beta}{\alpha(1+\delta)}},$$
$$c = \mp\delta\sqrt{\frac{\beta}{\alpha(1+\delta)}} \tag{2.5}$$

$$\lambda = \mp(1+\delta-r)\sqrt{\frac{\alpha\beta}{1+\delta}}, \qquad b = \pm\delta\sqrt{\frac{\beta}{\alpha(1+\delta)}},$$
$$c = \mp\delta r\sqrt{\frac{\beta}{\alpha(1+\delta)}} \tag{2.6}$$

Considering (2.2), (2.5), and (2.6), we know that

$$u = v^{1/\delta} = \left\{\frac{1}{2} \mp \frac{1}{2}\tanh\left[\frac{\delta}{2}\sqrt{\frac{\beta}{\alpha(1+\delta)}} \times \left(x \pm ((1+\delta)r - 1)\sqrt{\frac{\alpha\beta}{1+\delta}}\,t - \xi_0\right)\right]\right\}^{1/\delta} \tag{2.7}$$

$$u = v^{1/\delta} = \left\{\frac{1}{2} \mp \frac{1}{2}\coth\left[\frac{\delta}{2}\sqrt{\frac{\beta}{\alpha(1+\delta)}}\right.\right.$$

$$\times\left(x \pm ((1+\delta)r - 1)\sqrt{\frac{\alpha\beta}{1+\delta}}\,t - \xi_0\right)\Bigg]\Bigg\}^{1/\delta} \tag{2.8}$$

$$u = v^{1/\delta} = \left\{\frac{r}{2} \mp \frac{r}{2}\tanh\left[\frac{\delta r}{2}\sqrt{\frac{\beta}{\alpha(1+\delta)}}\right.\right.$$
$$\left.\left.\times\left(x \pm (1+\delta-r)\sqrt{\frac{\alpha\beta}{1+\delta}}\,t - \xi_0\right)\right]\right\}^{1/\delta} \tag{2.9}$$

$$u = v^{1/\delta} = \left\{\frac{r}{2} \mp \frac{r}{2}\coth\left[\frac{\delta r}{2}\sqrt{\frac{\beta}{\alpha(1+\delta)}}\right.\right.$$
$$\left.\left.\times\left(x \pm (1+\delta-r)\sqrt{\frac{\alpha\beta}{1+\delta}}\,t - \xi_0\right)\right]\right\}^{1/\delta} \tag{2.10}$$

Here, (2.7) and (2.10) reproduce the solutions obtained by Yang (1994).

Example 3. The generalized Burgers–Fisher equation (X. Y. Wang, 1988; X. Y. Wang *et al.*, 1990; B. L. Lu *et al.*, 1993a):

$$u_t + \alpha u^\delta u_x - \frac{m}{u}u_x^2 - u_{xx} = \beta u(1 - u^\delta) \qquad (\delta > 0)$$

Considering its traveling wave solution, we have

$$-\lambda u' + \alpha u^\delta u' - \frac{m}{u}u'^2 - u'' = \beta u(1 - u^\delta) \tag{3.1}$$

Since (3.1) has both odd- and even-rank terms, we use (4) to make an ansatz. Here, we can assume the same ansatz as in (2.2). Proceeding as in Example 2, we know that $h = 1/\delta$ and

$$\alpha - hbm - (h+1)b = 0, \qquad \lambda hc + h^2c^2m + h^2c^2 + \beta = 0$$
$$-\lambda hb + \alpha hc - 2h^2mbc - h(2h+1)bc + \beta = 0 \tag{3.2}$$

Solving the algebraic equation (3.2), we can easily arrive at

$$b = \frac{\alpha\delta}{1+\delta+m}, \qquad c = -\frac{\alpha\delta}{1+\delta+m},$$
$$\lambda = \frac{\alpha(1+m)}{1+\delta+m} + \frac{\beta(1+\delta+m)}{\alpha} \tag{3.3}$$

By (2.2) and (3.3) we know that

$$u = v^h = \left\{\frac{1}{2} - \frac{1}{2}\tanh\left[\frac{\alpha\delta}{2(1+\delta+m)} \times \left(x - \left[\frac{\alpha(1+m)}{1+\delta+m} + \frac{\beta(1+\delta+m)}{\alpha}\right]t - \xi_0\right)\right]\right\}^{1/\delta} \tag{3.4}$$

$$u = v^h = \left\{\frac{1}{2} - \frac{1}{2}\coth\left[\frac{\alpha\delta}{2(1+\delta+m)} \times \left(x - \left[\frac{\alpha(1+m)}{1+\delta+m} + \frac{\beta(1+\delta+m)}{\alpha}\right]t - \xi_0\right)\right]\right\}^{1/\delta} \tag{3.5}$$

Here, (3.4) corresponds to the counterpart solutions obtained previously by X. Y. Wang (1988), X. Y. Wang *et al.* (1990), and B. L. Lu *et al.* (1993a).

Example 4. The generalized Burgers–Huxley equation (B. L. Lu *et al.*, 1993a):

$$u_t + \alpha u^\delta u_x - \frac{m}{u}u_x^2 - Du_{xx} = \beta u(1-u^\delta)(u^\delta - r) \qquad (\delta > 0, \quad D > 0)$$

Considering its traveling wave solution, we see

$$-\lambda u' + \alpha u^\delta u' - \frac{m}{u}u'^2 - Du'' = \beta u(1-u^\delta)(u^\delta - r) \tag{4.1}$$

Again, (4.1) has both odd- and even-rank terms. Therefore we try to use a type (4) ansatz. Here, we use the ansatz (2.2). Proceeding as before, we know that $h = 1/\delta$ and

$$-\lambda hc - mh^2c^2 - Dh^2c^2 + \beta r = 0,$$
$$\alpha hb - mh^2b^2 - Db^2(h+1)h + \beta = 0$$
$$-\lambda hb + \alpha hc - 2mh^2bc - D(2h+1)hbc - (r+1)\beta = 0$$

which yield

$$b_{1\pm} = \frac{\delta(\alpha \pm y)}{2(D(1+\delta)+m)}, \qquad c_{1\pm} = -\frac{\delta(\alpha \pm y)}{2(D(1+\delta)+m)}$$

$$\lambda_{1\pm} = \frac{\alpha(m(1+r) + D(1+r+\delta r)) \pm [m(1-r) + D(1-(1+\delta)r)]y}{2(D(1+\delta)+m)} \tag{4.2}$$

$$b_{2\pm} = \frac{\delta(\alpha \pm y)}{2(D(1+\delta)+m)}, \qquad c_{2\pm} = -\frac{r\delta(\alpha \pm y)}{2(D(1+\delta)+m)} \tag{4.3}$$

$$\lambda_{2\pm} = \frac{\alpha(m(1+r)+D(1+r+\delta)) \pm [m(r-1)-D\,(1+\delta-r)]y}{2(D(1+\delta)+m)}$$

where $y = \sqrt{\alpha^2 + 4\beta[D(1+\delta)+m]}$. From (4.2)–(4.3), it follows that

$$u = \left\{\frac{1}{2} - \frac{1}{2}\tanh\left[\frac{\delta(\alpha \pm y)}{4(D(1+\delta)+m)}(x - \lambda_{1\pm}t - \xi_0)\right]\right\}^{1/\delta}$$

or

$$u = \left\{\frac{1}{2} - \frac{1}{2}\coth\left[\frac{\delta(\alpha \pm y)}{4(D(1+\delta)+m)}(x - \lambda_{1\pm}t - \xi_0)\right]\right\}^{1/\delta} \qquad (4.4)$$

and

$$u = \left\{\frac{r}{2} - \frac{r}{2}\tanh\left[\frac{r\delta(\alpha \pm y)}{4(D(1+\delta)+m)}(x - \lambda_{2\pm}t - \xi_0)\right]\right\}^{1/\delta}$$

or

$$u = \left\{\frac{r}{2} - \frac{r}{2}\coth\left[\frac{r\delta(\alpha \pm y)}{4(D(1+\delta)+m)}(x - \lambda_{2\pm}t - \xi_0)\right]\right\}^{1/\delta} \qquad (4.5)$$

with ξ_0 arbitary and y, $\lambda_{j\pm}$ $(j = 1, 2)$ as mentioned above. Here, when $D = 1$ we reproduce the smooth solutions obtained in B. L. Lu *et al.* (1993a) with additional singular ones. It should be pointed out that there are two sign errors in the expression for solitary wave solutions given in that work.

Example 5. Consider the MKdV–Burgers equation (M. L. Wang, 1996; Zhang, 1996):

$$u_t + u^2 u_x + \alpha u_{xx} + \beta u_{xxx} = 0$$

Assume that $u(-\infty) = 0$ for simplicity and insert $u = u(\xi)$, $\xi = x - \lambda t$, to obtain

$$-\lambda u + \frac{1}{3}u^3 + \alpha u' + \beta u'' = 0 \qquad (5.1)$$

Obviously, Eq. (5.1) have both even- and odd-order terms. This suggests we use an ansatz of form (4). To this end, set

$$u' = bu^2 + cu \qquad (5.2)$$

where b, c are undetermined constants. Then,

$$u'' = 2b^2u^3 + 3bcu^2 + c^2\,u \qquad (5.3)$$

Substitute (5.2) and (5.3) into (5.1) and compare the same powers of u to get

$$-\lambda + \alpha c + \beta c^2 = 0, \qquad 2\beta b^2 + \frac{1}{3} + 0, \qquad \alpha b + 3\beta bc = 0 \quad (5.4)$$

which gives

$$b = \pm\sqrt{-\frac{1}{6\beta}}, \qquad c = -\frac{\alpha}{3\beta}, \qquad \lambda = -\frac{2\alpha^2}{9\beta} \quad (5.5)$$

It is noted that (5.2) has the solutions

$$u = -\frac{c}{2b}\tanh\left[\frac{c}{2}(\xi - \xi_0)\right] - \frac{c}{2b} \quad \text{or} \quad -\frac{c}{2b}\coth\left[\frac{c}{2}(\xi - \xi_0)\right] - \frac{c}{2b} \quad (5.6)$$

with ξ_0 an arbitary constant.

Considering (5.2), (5.5), and (5.6), we have

$$u = \pm\frac{\alpha}{\sqrt{-6\beta}}\left\{1 + \tanh\left[-\frac{\alpha}{6\beta}\left(\left(x + \frac{2\alpha^2}{9\beta}t\right) - \xi_0\right)\right]\right\} \quad (5.7)$$

or

$$u = \pm\frac{\alpha}{\sqrt{-6\beta}}\left\{1 + \coth\left[-\frac{\alpha}{6\beta}\left(\left(x + \frac{2\alpha^2}{9\beta}t\right) - \xi_0\right)\right]\right\} \quad (5.8)$$

Here, by a different method, (5.7) gives the counterparts of M. L. Wang (1996) and Zhang (1996). Meanwhile, (5.8) yields new, additional singular solutions. Furthermore, we can easily obtain explicit solutions to other equations of M. L. Wang (1996) and Zhang (1996) following the exploratory approach here.

Example 6. The approximate equations for long water waves (Kupershmidt, 1985; M. L. Wang *et al.*, 1996):

$$u_t - uu_x - v_x + \tfrac{1}{2}u_{xx} = 0$$
$$v_t - (uv)_x - \tfrac{1}{2}v_{xx} = 0$$

For this coupled equations, let us assume that the relation

$$v = A + Bu + Cu^2 \quad (6.1)$$

holds between the two unknown u and v. Consider its traveling wave solutions $u = u(\xi)$, $v = v(\xi)$, $\xi = x - \lambda t$. For simplicity, suppose that $u(-\infty) = v(-\infty)$

$=0$, that is, $A = 0$. Then, integrating the coupled equations once leads to

$$-\lambda u - \frac{1}{2}u^2 - v + \frac{1}{2}u' = 0, \qquad -\lambda v - uv - \frac{1}{2}v' = 0 \tag{6.2}$$

Insert (6.1) into (6.2) to obtain

$$u' = (1 + 2C)u^2 + 2(\lambda + B)u \tag{6.3}$$

and

$$-\lambda(Bu + Cu^2) - u(Bu + Cu^2) - \frac{1}{2}(B + 2Cu)u' = 0 \tag{6.4}$$

Substitute (6.3) into (6.4) and balance the same powers of u in the resulting equation to obtain

$$B(2\lambda + B) = 0, \qquad \lambda C + B + 2C(\lambda + B) + (\tfrac{1}{2} + C)B = 0,$$
$$C + C(2C + 1) = 0$$

which gives the following solution of physical interest:

$$\lambda = -B/2, \qquad C = -1 \tag{6.5}$$

Thus, from (6.3) and (6.5) it follows that

$$u' = -u^2 + Bu \tag{6.6}$$

By (6.1) and (6.6) it follows that

$$u = \frac{B}{2}\left[1 + \tanh\left(\frac{B}{2}\left(x + \frac{B}{2}t - \xi_0\right)\right)\right],$$
$$v = \frac{B^2}{4}\operatorname{sech}^2\left(\frac{B}{2}\left(x + \frac{B}{2}t - \xi_0\right)\right) \tag{6.7}$$

and

$$u = \frac{B}{2}\left[1 + \coth\left(\frac{B}{2}\left(x + \frac{B}{2}t - \xi_0\right)\right)\right],$$
$$v = -\frac{B^2}{4}\operatorname{csch}^2\left(\frac{B}{2}\left(x + \frac{B}{2}t - \xi_0\right)\right) \tag{6.8}$$

Here, by a very simple method (6.7) reproduces the counterpart of M. L. Wang *et al.* (1996).

Example 7. The dispersive long-wave equations (Lou, 1993; M. L. Wang *et al.*, 1996):

$$u_{yt} + \eta_{xx} + \tfrac{1}{2}(u^2)_{yy} = 0$$
$$\eta_t + (u\eta + u + u_{xy})_x = 0$$

These coupled equations become

$$u_t + \eta_z + \tfrac{1}{2}(u^2)_k = 0, \qquad \eta_t + (u\eta + u_{zz})_z = 0 \tag{7.1}$$

after setting $u = u(z, t) = u(x + y, t)$ and $\eta = \eta(z, t) = \eta(x + y, t)$. Here, the traveling wave solutions of the form $u(z, t) = u(\xi)$, $\eta(z, t) = \eta(\xi)$, $\xi = z - \lambda t$, are considered. Assume that

$$\eta = A + Bu + Cu^2, \qquad u(-\infty) = 0 \tag{7.2}$$

Insert (7.2) into the first equation of (7.1) and balance the same powers of u by integrating once to give

$$\lambda = B, \qquad C = -\tfrac{1}{2} \tag{7.3}$$

Putting (7.2) into the second equation of (7.1) and integrating once, we obtain

$$u'' = (\lambda B - A)u + (\lambda C - B)u^2 + \tfrac{1}{2}u^3$$

from which it follows that

$$u'^2 = (B^2 - A)u^2 - Bu^3 + \tfrac{1}{4}u^4 \tag{7.4}$$

Solve (7.4) to get

$$u = \frac{6B(B^2 - A)\exp[\pm\sqrt{B^2 - A}\,(x - Bt - \xi_0)]}{B^2(1 + \exp[\pm\sqrt{B^2 - A}\,(x - Bt - \xi_0)])^2 - B^2 + A} \tag{7.5}$$

where $B^2 - A > 0$. Obviously, (7.5) is a smooth solution when $A > 0$, and a singular one when $A < 0$. Taking into account of the ansatz (7.2), we deduce that

$$\eta = A + Bu - \tfrac{1}{2}u^2 \tag{7.6}$$

with u given by (7.5). Our result is more general than that of M. L. Wang *et al.* (1996).

Example 8. The coupled KdV equations (Guha-Roy, 1987; B. L. Lu *et al.*, 1993b):

$$u_t + avv_x + buu_x + ru_{xxx} = 0$$
$$u_t + d(uv)_x + evv_x = 0$$

Considering the traveling wave solution $u = u(\xi)$, $v = v(\xi)$, $\xi = x - \lambda t$, let us assume that

$$u = A + Bv + Cv^2, \qquad v(-\infty) = 0 \tag{8.1}$$

Substitute (8.1) into the second equation of the coupled system, integrate once, and balance the same powers of v in the resulting equation to give

$$C = 0, \qquad \lambda = Ad, \qquad B = -\frac{e}{2d} \tag{8.2}$$

Insert $u = A + Bv$ into the first equation of the coupled system to give

$$v'' = -\frac{1}{rB}\left[B(Ab - \lambda)v + \tfrac{1}{2}(a + B^2 b)v^2\right] \tag{8.3}$$

Solving (8.3) and taking into account of (8.1) and (8.2), we have

$$\begin{aligned} v &= \frac{6Ade(b-d)}{4ad^2 + be^2}\operatorname{sech}^2\left[\sqrt{\frac{A(d-b)}{4r}}(x - Adt - \xi_0)\right] \\ u &= A + Bv = A - \frac{3Ae^2(b-d)}{4ad^2 + be^2}\operatorname{sech}^2\left[\sqrt{\frac{A(d-b)}{4r}}(x - Adt - \xi_0)\right] \end{aligned} \tag{8.4}$$

and

$$\begin{aligned} v &= -\frac{6Ade(b-d)}{4ad^2 + be^2}\operatorname{csch}^2\left[\sqrt{\frac{A(d-b)}{4r}}(x - Adt - \xi_0)\right] \\ u &= A + \frac{3Ae^2(b-d)}{4ad^2 + be^2}\operatorname{csch}^2\left[\sqrt{\frac{A(d-b)}{4r}}(x - Adt - \xi_0)\right] \end{aligned} \tag{8.5}$$

Obviously, (8.4) yields the counterpart of B. L. Lu *et al.* (1993b).

Example 9. Another kind of coupled KdV equation (Guha-Roy, 1987; B. L. Lu *et al.*, 1993b):

$$\begin{aligned} u_t + av^2v_x + bu^2u_x + cuu_x + ru_{xxx} &= 0 \\ v_t + d(uv)_x + evv_x &= 0 \end{aligned}$$

Proceed as in Example 8 to make the ansatz

$$u = A + Bv, \qquad v(-\infty) = 0 \tag{9.1}$$

Using (9.1) and considering the second equation of the coupled system, we have

$$-\lambda v + d(A + Bv)\, v + \frac{e}{2} v^2 = 0$$

from which it follows that

$$\lambda = Ad, \qquad B = -\frac{e}{2d} \tag{9.2}$$

Combining (9.1), (9.2) with the first equation of the coupled system and integrating once, we see that

$$v'' = a_1 v + a_2 v^2 + a_3 v^3 \tag{9.3}$$

where

$$a_1 = -\frac{A}{r}(Ab - d + c), \qquad a_2 = \frac{e}{2dr}\left(Ab + \frac{c}{2}\right), \qquad a_3 = \frac{be^3 - 8ad^3}{24Bd^3 r} \tag{9.4}$$

Solve (9.3) to find

$$v = \frac{12a_1a_2 \exp[\pm\sqrt{a_1}(x - Adt - \xi_0)]}{9a_1a_3 - 2a_2^2(1 + \exp[\pm\sqrt{a_1}(x - Adt - \xi_0)])^2} \tag{9.5}$$

for $a_2 \neq 0$, $a_1 > 0$. When $a_2 = 0$, that is, $e = 0$ or $Ab + c/2 = 0$,

$$v = \pm\sqrt{-\frac{2}{a_3}}\,\operatorname{sech}[\sqrt{a_1}(x - Adt - \xi_0)], \qquad a_3 < 0 \tag{9.6}$$

or

$$v = \pm\sqrt{\frac{2}{a_3}}\,\operatorname{csch}[\sqrt{a_1}(x - Adt - \xi_0)], \qquad a_3 > 0 \tag{9.7}$$

Thus we have three sets of solutions (9.5), (9.6), and (9.7) for v and their corresponding u given by

$$u = A - \frac{e}{2d} v \tag{9.8}$$

with a_1, a_2, and a_3 expressed by (9.4). Obviously, when $9a_1a_3 - 2a_2^2 < 0$, (9.5) is a smooth solution; otherwise it stands for a singular solution. Therefore, by a different method we can find more solution, than in B. L. Lu *et al*. (1993b).

4. CONCLUSIONS

We have succeeded in finding exact and explicit traveling wave solutions to some nonlinear evolution equations according to our exploratory approach as stated in the Introduction. All the algebraic calculations in the present paper have been verified by using *Mathematica*. Some of these solutions were first given by using other methods. Compared with previously used methods, ours is direct and easy to realize and can yield richer solutions. It goes without saying that we cannot exhaust all the existing explicit solutions to a given NLEPDE even if we manage to find a suitable ansatz by this exploratory approach. However, this exploratory approach is not only generally effective, but also concise. We can use the approach to many other nonlinear equations or coupled ones, which will be published elsewhere.

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REFERENCES

Coffey, M. W. (1990). *SIAM Journal of Applied Mathematics*, **50**, 1580.
Coffey, M. W. (1992). *SIAM Journal of Applied Mathematics*, **52**, 929.
Feng, X. (1996). *Physics Letters A*, **213**, 167.
Feng, X. (1998). *Journal of Jishou University*, **19**, 6.
Feng, X. (1999a). *Journal of Physics A*, submitted.
Feng, X. (1999b). *Physics Letters A*, submitted.
Guha-Roy, C. (1987). *Journal of Mathematical Physics*, **28**, 2089.
Hereman, W., and Takaoka, M. (1990). *Journal of Physics A*, **23**, 4805.
Hereman, W., Banerjee, P. P., Korpel, A., Assanto, G., Van Immerzeele, A., and Meerpoel, A. (1986). *Journal of Physics A*, **19**, 607.
Huang, G., Luo, S., and Dai, X. (1989). *Physics Letters A*, **139**, 373.
Ito, M. (1980). *Journal of Physics Society of Japan*, **49**, 771.
Kupershmidt, B. A. (1985). *Communications in Mathematical Physics*, **99**, 51.
Lan, H., and Wang, K. (1990), *Journal of Physics A*, **23**, 4097.
Lou, S. Y. (1993). *Physics Letters A*, **176**, 96.
Lu, B. L., Xiu, B. Z., Pang, Z. L., and Jiang, X. F. (1993a). *Physics Letters A*, **175**, 113.
Lu, B. L., Xiu, B. Z., Pang, Z. L., and Jiang, X. F.(1993b). *Physics Letters A*, **180**, 61.
Lu, B. Q. (1994). *Physics Letters A*, **189**, 25.
Parkes, E. J., Zhu, Z., Duffy, B. R., and Huang, H. C. (1998). *Physics Letters A*, **248**, 219.
Sachev, P. L. (1987). *Nonlinear Diffusive Waves*, Cambridge University Press, Cambridge.
Wang, M. L. (1996). *Physics Letters A*, **213**, 279.
Wang, M. L., Li, Z., and Zhou, Y. (1996). *Physics Letters A*, **216**, 67.

Wang, M. X., Xiong, S. L., and Ye, Q. X. (1994). *Journal of Mathematical Analysis and its Applications*, **182**, 705.
Wang, X. Y. (1988). *Physics Letters A*, **131**, 277.
Wang, X. Y., Chu, Z. S., and Lu, Y. K. (1990). *Journal of Physics A*, **23**, 271.
Yang, Z. J. (1994). *Journal of Physics A*, **27**, 2837.
Yang, Z. J. (1995). *International Journal of Theoretical Physics*, **34**, 641.
Zhang, J. F. (1996). *International Journal of Theoretical Physics*, **35**, 1793.